



The Restricted Detour Polynomial of a Ladder Graph L_n^*

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Abstract

The restricted detour distance $D^*(u, v)$ between two vertices u and v of a connected graph G is the length of a longest $u - v$ path P in G such that $\langle V(P) \rangle = P$. The restricted detour polynomial of a graph G denoted by $D^*(G; x)$ and defined as follows:

$$D^*(G; x) = \sum_{u,v} x^{D^*(u,v)},$$

where the summation is taken over all unordered pairs u, v of vertices of G . The **restricted detour polynomials** and **restricted detour indices** of the ladder graph L_n^* are obtained in this paper.

Introduction

The concept of Hosoya polynomial was first forward in 1988 by Hosoya [1]. Several authors have obtained Hosoya polynomials for special graphs, graphs having some kind of regularity and for compound graphs obtained.

In this paper, we consider finite connected graphs without loops or multiple edges. For undefined concepts and notations, see [7] and [9].

Let G be a connected graph, and let u and v be any two vertices of G . The (standard) **distance** $d(u, v)$ between u and v in G is the length of a shortest $u - v$ path P in G [9].

It is clear that the induced subgraph $\langle V(P) \rangle$ is P itself. Based on this observation, Chartrand, et al [6], in 1993 defined the detour distance $d^*(u, v)$ between u and v as the length of a longest $u - v$ path P for which $\langle V(P) \rangle = P$.

Later on, Chartrand, et al and many authors [8], defined the concepts of **detour distance** $D(u, v)$ between vertices u and v in G , as the length of a longest $u - v$ path P , without assuming the induced condition $\langle V(P) \rangle = P$. Therefore to differentiate between the two concepts, the **restricted detour distance** between u and v , denoted by $D_G^*(u, v)$ or simply $D^*(u, v)$ [5].

From the definition of the concept D^* on the vertex set $V(G)$, it is clear that $D^*(u, v) = 0$ if and only if $u=v$ and $D^*(u, v) = 1$ if and only if uv is an edge of G . One can notice that the restricted detour distance is not metric on $V(G)$ [3].

An induced $u-v$ path of length $D^*(u, v)$ will be called a **restricted detour path** [6], moreover a connected graph G is known as a **restricted detour graph** if $d(u, v) = D^*(u, v)$ for all vertices u and v of G . All trees and

all complete graphs are **restricted detour graphs**. However, cycle graph of order $n \geq 5$ is not restricted detour. For more properties and results on restricted detour distances, one may see [6].

Restricted Detour Polynomials

Let G be a connected graph. The concept of Hosoya polynomial $H(G; x)$ is based on standard distance, (see [1], [4] and [10]), and the concept of detour polynomials $D(G; x)$ is based on detour distance. In a similar way, the concept of **restricted detour polynomial**, denoted by $D^*(G; x)$ is defined as

$$D^*(G; x) = \sum_{u,v} x^{D^*(u,v)},$$

where the summation is taken over all unordered pairs u, v of vertices of G .

The **index** of G with respect to **restricted detour distance** is denoted by $dd^*(G)$ and defined by

$$dd^*(G) = \sum_{u,v} D_G^*(u, v),$$

and will be called **restricted detour index** of G . It is clear that $dd^*(G) = \frac{d}{dx} D^*(G; x)|_{x=1}$.

One can easily notice that

$$D^*(G; x) = \sum_{k \geq 0} C^*(G, k)x^k.$$

In which $C^*(G; k)$ is the number of unordered pairs of vertices u, v of G such that $D^*(u, v) = k$.

Let u be any vertex of G , and let $C^*(u, G; k)$ be the number of vertices of v of G such that $D^*(u, v) = k$.

Then the polynomial is defined by

$$D^*(u, G; x) = \sum_{k \geq 0} C^*(u, G; k)x^k,$$

is called the **restricted detour polynomial of vertex u** . It is clear that

$$D^*(G; x) = \frac{1}{2} \left(\sum_{k \geq 0} D^*(u, G; x)x^k + P \right).$$

These concepts are illustrated in the following example.

Example 1: Consider the graph G shown in Figure 1.

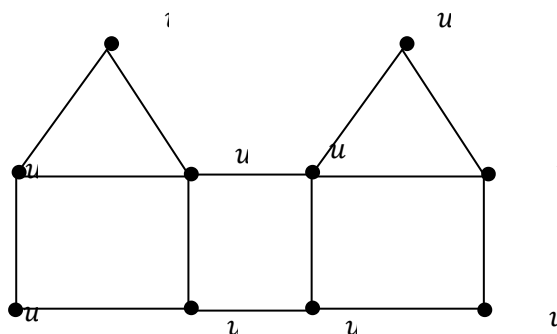


Figure 1. The graph G

By direct calculation using Figure 1, we obtain the restricted detour distance of the vertices, thus

$$\begin{aligned} C^*(G, 0) &= 10, & C^*(G, 1) &= 14, & C^*(G, 2) &= 6, & C^*(G, 3) &= 7, \\ C^*(G, 4) &= 8, & C^*(G, 5) &= 7, & C^*(G, 6) &= 2, & C^*(G, 7) &= 1. \end{aligned}$$

Hence, the restricted detour polynomial of G is

$$D^*(G; x) = 10 + 14x + 6x^2 + 7x^3 + 8x^4 + 7x^5 + 2x^6 + x^7 \text{ and } dd^*(G) = 143.$$

The restricted detour polynomials and indices of some compound and special graphs are obtained by [2, 3, 5 and 10].

In this paper, we obtain the **restricted detour polynomial** of the Ladder graph L_n^* .

The Restricted Detour Polynomial of a Ladder graph L_n^*

Let P_n be a path of order n , $n \geq 2$. The Ladder graph L_n is $k_2 \times P_n$. It is clear that $p(L_n) = 2n$, $q(L_n) = 3n - 2$, and $diam L_n = n$. It is known [4], that the Hosoya polynomial of P is given by

$$H(P_n; x) = \sum_{k=0}^{n-1} (n - k)x^k.$$

A Ladder [5], L_n^* is L_n with edge $u'_1 u_n$ as in Figure 2. It is clear that $p(L_n^*) = 2n$, $q(L_n^*) = 3n - 1$, $diam L_n^* = \lfloor \frac{n}{2} \rfloor + 1$ and $Diam L_n^* = 2n - 1$.

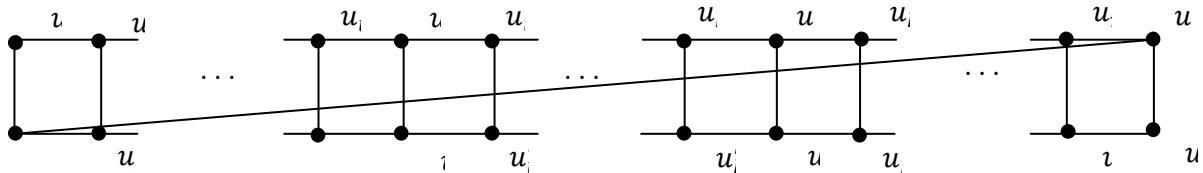


Figure 2. The Ladder graph L_n^* , $n \geq 2$

The distance between vertices u_i and u_{i+k} , also the distance between u_i and u'_{i+k} in L_n^* are obtained in [5] and given by the following proposition.

Proposition 1: For $k \geq 0$,

$$D_{L_n^*}^*(u_i, u_{i+k}) = k + 2 \lfloor \frac{k-1}{4} \rfloor,$$

$$D_{L_n^*}^*(u_i, u'_{i+k}) = k + 1 + 2 \lfloor \frac{k}{4} \rfloor. \blacksquare$$

Using this fact, the restricted detour polynomials of the ladder graph L_n^* are obtained in the next two successive theorems.

Theorem 2. For odd n , $n \geq 7$, we have

$$\begin{aligned} D^*(L_n^*; x) = & 2n + (3n - 1)x + 2x^2 + 2 \sum_{k=\frac{n+1}{2}+1}^{n-1} x^{k-1+2\lfloor \frac{k-2}{4} \rfloor} + 2x^{n-1+2\lfloor \frac{n-3}{4} \rfloor} + 2 \sum_{k=\frac{n+1}{2}+1}^{n-1} x^{k+2\lfloor \frac{k-1}{4} \rfloor} + \\ & x^{n+2\lfloor \frac{n-2}{4} \rfloor} + 2 \sum_{i=1}^{\frac{n+1}{2}} \sum_{k=i+2}^{\frac{n+1}{2}+(i-1)} x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k-1}{4} \rfloor+1} + \sum_{i=1}^{\frac{n+1}{2}} \sum_{k=i+2}^{\frac{n+1}{2}+(i-1)} x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k}{4} \rfloor+2} + \\ & 2 \sum_{i=2}^{\frac{n+1}{2}-1} \sum_{k=\frac{n+1}{2}+i}^n x^{(k-i)+2\lfloor \frac{k-i-1}{4} \rfloor} + 2 \sum_{i=2}^{\frac{n+1}{2}-1} \sum_{k=\frac{n+1}{2}+i}^n x^{(k-i)+1+2\lfloor \frac{k-i}{4} \rfloor} + \\ & 2 \sum_{i=\frac{n+1}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k-1}{4} \rfloor+1} + \sum_{i=\frac{n+1}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k}{4} \rfloor+2} + \\ & \sum_{i=2}^{n-2} x^{n+2\lfloor \frac{i-2}{4} \rfloor+2\lfloor \frac{n-i-2}{4} \rfloor+1} + \sum_{i=1}^{n-1} x^{(n-1)+2\lfloor \frac{i-3}{4} \rfloor+2\lfloor \frac{n-i-3}{4} \rfloor} + \sum_{k=1}^{\frac{n+1}{2}} \sum_{i=k+2}^{\frac{n+1}{2}+(k-1)} x^{k+2\lfloor \frac{k-2}{4} \rfloor+(n-i)+2\lfloor \frac{n-i-1}{4} \rfloor} + \\ & \sum_{k=\frac{n+1}{2}+1}^{n-2} \sum_{i=k+2}^n x^{k+2\lfloor \frac{k-2}{4} \rfloor+(n-i)+2\lfloor \frac{n-i-1}{4} \rfloor}. \end{aligned}$$

Proof.

Let u_i and u_k (or u'_k) where $|i - k| \neq 1$ be two vertices of L_n^* . We refer to Figure 2, the following cases can be distinguished

Case I : For $i = 1$ and $\frac{n+1}{2} + 1 \leq k \leq n - 1$

(1) Notice that the path P_1 ,

$P_1 : u_1, u'_1, u'_2, u'_3, u_3, u_4, u_5, \dots, u'_k, u_k$ (or \dots, u_{k-1}, u_k)

is the longest $u_1 - u_k$ path with $\langle P_1 \rangle = P_1$.

Evidently by Proposition 1, $D^*(u_1, u_k) = (k - 1) + 2 \lfloor \frac{k-2}{4} \rfloor$.

From the symmetry of L_n^* , we have $D^*(u_1, u_k) = D^*(u'_1, u'_k)$. Thus the corresponding polynomial is

$$D_1(x) = 2 \sum_{k=\frac{n+1}{2}+1}^{n-1} x^{(k-1)+2\lfloor\frac{k-2}{4}\rfloor}.$$

(2) Notice that the path P_2 ,

$P_2 : u_1, u'_1, u'_2, u'_3, u_3, u_4, u_5, \dots, u_k, u'_k$ (or \dots, u'_{k-1}, u'_k)

is the longest $u_1 - u'_k$ path with $\langle P_2 \rangle = P_2$ which of the length $k + 2 \lfloor \frac{k-1}{4} \rfloor$ [as in Proposition 1].

Evidently $D^*(u_1, u'_k) = D^*(u'_1, u_k)$.

In this fact the corresponding polynomial is

$$D_2(x) = 2 \sum_{k=\frac{n+1}{2}+1}^{n-1} x^{k+2\lfloor\frac{k-1}{4}\rfloor}.$$

Case II : For $i = 1$ and $k = n$

(1) For the vertices u_1 and u_n there is a restricted detour path P_3 , namely

$P_3 : u_1, u_2, u'_2, u'_3, u'_4, u_4, u_5, u_6, \dots, u'_n, u_n$ (or \dots, u_{n-1}, u_n)

which is of the length $(n - 1) + 2 \lfloor \frac{n-3}{4} \rfloor$.

Now, $D^*(u_1, u_n) = D^*(u'_1, u'_n)$, this produce the polynomial

$$D_3(x) = 2x^{(n-1)+2\lfloor\frac{n-3}{4}\rfloor}.$$

(2) For the vertices u_1 and u'_n there is a restricted detour path P_4 , namely

$P_4 : u_1, u_2, u'_2, u'_3, u'_4, u_4, u_5, u_6, \dots, u_n, u'_n$ (or \dots, u'_{n-1}, u'_n)

which is of the length $n + 2 \lfloor \frac{n-2}{4} \rfloor$ the corresponding polynomial is

$$D_4(x) = x^{n+2\lfloor\frac{n-2}{4}\rfloor}.$$

(3) For particular if $k = 2$, notice that there is exactly two restricted detour paths between u_1 and u'_2 which are $P_5 : u_1, u_2, u'_2$ or $P'_5 : u_1, u'_1, u'_2$ both are of length 2. Evidently $D^*(u_1, u'_2) = D^*(u_{n-1}, u'_n)$, the corresponding polynomial is

$$D_5(x) = 2x^2.$$

Case III : For $1 \leq i \leq \frac{n+1}{2}$ and $i + 2 \leq k \leq \frac{n+1}{2} + (i - 1)$

(1) From the Figure 2. We notice that the path P_6 ,

$P_6 : u_i, u'_i, u'_{i-1}, u'_{i-2}, u_{i-2}, u_{i-3}, u_{i-4}, \dots, u'_2, u'_1$ (or \dots, u_1, u'_1), $u_n, u'_n, u'_{n-1}, u'_{n-2}, u_{n-2}, u_{n-3}, u_{n-4}, \dots, u'_k, u_k$ (or \dots, u_{k+1}, u_k) is a longest $u_i - u_k$ path with $\langle P_6 \rangle = P_6$ and has length

$$i + 2 \lfloor \frac{i-1}{4} \rfloor + (n - k) + 2 \lfloor \frac{n-k-1}{4} \rfloor + 1.$$

Evidently, $D^*(u_i, u_k) = D^*(u'_{n-i+1}, u'_{n-k+1})$, this produce the polynomial

$$D_6(x) = 2 \sum_{i=1}^{\frac{n+1}{2}} \sum_{k=i+2}^{\frac{n+1}{2}+(i-1)} x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k-1}{4}\rfloor+1}.$$

(2) Notice that the restricted detour path between the vertices u_i and u'_k is P_7 , where

$P_7 : u_i, u'_i, u'_{i-1}, u'_{i-2}, u_{i-2}, u_{i-3}, u_{i-4}, \dots, u'_2, u'_1$ (or \dots, u_1, u'_1), $u_n, u'_n, u'_{n-1}, u'_{n-2}, u_{n-2}, u_{n-3}, u_{n-4}, \dots, u_k, u'_k$ (or \dots, u'_{k+1}, u'_k)

which is of the length $i + 2 \lfloor \frac{i-1}{4} \rfloor + (n - k) + 2 \lfloor \frac{n-k}{4} \rfloor + 2$.

Now the corresponding polynomial is

$$D_7(x) = \sum_{i=1}^{\frac{n+1}{2}} \sum_{k=i+2}^{\frac{n+1}{2}+(i-1)} x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k}{4}\rfloor+2}.$$

Case IV : For $2 \leq i \leq \frac{n+1}{2} - 1$ and $\frac{n+1}{2} + i \leq k \leq n$

(1) We notice that the restricted detour path between the vertices u_i and u_k , namely $P_8 : u_i, u'_i, u'_{i+1}, u'_{i+2}, u_{i+2}, u_{i+3}, u_{i+4}, \dots, u'_k, u_k$ (or \dots, u'_{k-1}, u_k) with $\langle P_8 \rangle = P_8$ which is of the length $(k - i) + 2 \left\lfloor \frac{k-i-1}{4} \right\rfloor$.

Evidently $D^*(u_i, u_k) = D^*(u'_i, u'_k)$, the corresponding polynomial is

$$D_8(x) = 2 \sum_{i=2}^{\frac{n+1}{2}-1} \sum_{k=\frac{n+1}{2}+i}^n x^{(k-i)+2 \left\lfloor \frac{k-i-1}{4} \right\rfloor}.$$

(2) There is the restricted detour path between the vertices u_i and u'_k which of the length $(k - i) + 1 + 2 \left\lfloor \frac{k-i}{4} \right\rfloor$. For the symmetry of L_n^* , we have $D^*(u_i, u'_k) = D^*(u'_i, u_k)$, by this fact the polynomial is

$$D_9(x) = 2 \sum_{i=2}^{\frac{n+1}{2}-1} \sum_{k=\frac{n+1}{2}+i}^n x^{(k-i)+1+2 \left\lfloor \frac{k-i}{4} \right\rfloor}.$$

Case V : For $\frac{n+1}{2} + 1 \leq i \leq n - 2$ and $i + 2 \leq k \leq n$

(1) For the values of i and k , the restricted detour path between the vertices u_i and u_k is of the length $i + 2 \left\lfloor \frac{i-1}{4} \right\rfloor + (n - k) + 2 \left\lfloor \frac{n-k-1}{4} \right\rfloor + 1$.

Evidently, $D^*(u_i, u_k) = D^*(u'_{n-i+1}, u'_{n-k+1})$, and it is corresponding polynomial is

$$D_{10}(x) = 2 \sum_{i=\frac{n+1}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2 \left\lfloor \frac{i-1}{4} \right\rfloor + (n-k) + 2 \left\lfloor \frac{n-k-1}{4} \right\rfloor + 1}.$$

(2) For the same values of i and k the restricted detour path between the vertices u_i and u'_k , is of the length

$i + 2 \left\lfloor \frac{i-1}{4} \right\rfloor + (n - k) + 2 \left\lfloor \frac{n-k}{4} \right\rfloor + 2$. the corresponding polynomial is

$$D_{11}(x) = \sum_{i=\frac{n+1}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2 \left\lfloor \frac{i-1}{4} \right\rfloor + (n-k) + 2 \left\lfloor \frac{n-k}{4} \right\rfloor + 2}.$$

Case VI :

For $2 \leq i \leq n - 2$.

We notice that from Figure 2, there is a restricted detour path between the vertices u_i and u'_{i+1} , of length $i + 2 \left\lfloor \frac{i-2}{4} \right\rfloor + (n - i) + 2 \left\lfloor \frac{n-i-2}{4} \right\rfloor + 1$.

Also, between vertices u'_i and u_{i+1} there is a restricted detour path of length $(i - 1) + 2 \left\lfloor \frac{i-3}{4} \right\rfloor + (n - i) + 2 \left\lfloor \frac{n-i-3}{4} \right\rfloor$, their corresponding polynomials are

$$D_{12}(x) = \sum_{i=2}^{n-2} x^{n+2 \left\lfloor \frac{i-2}{4} \right\rfloor + 2 \left\lfloor \frac{n-i-2}{4} \right\rfloor + 1},$$

and

$$D_{13}(x) = \sum_{i=1}^{n-1} x^{(n-1)+2 \left\lfloor \frac{i-3}{4} \right\rfloor + 2 \left\lfloor \frac{n-i-3}{4} \right\rfloor}.$$

Case VII :

(1) For $1 \leq k \leq \frac{n+1}{2}$ and $+2 \leq i \leq \frac{n+1}{2} + (k - 1)$. The restricted detour path between the vertices u'_k and u_i is of the length $k + 2 \left\lfloor \frac{k-2}{4} \right\rfloor + (n - i) + 2 \left\lfloor \frac{n-i-1}{4} \right\rfloor$, the corresponding polynomial is

$$D_{14}(x) = \sum_{k=1}^{\frac{n+1}{2}} \sum_{i=k+2}^{\frac{n+1}{2}+(k-1)} x^{k+2\lfloor\frac{k-2}{4}\rfloor+(n-i)+2\lfloor\frac{n-i-1}{4}\rfloor}.$$

(2) For $\frac{n+1}{2} \leq k \leq n - 2$ and $k + 2 \leq i \leq n$. The restricted detour path between the vertices u'_k and u_i is of the length $k + 2\lfloor\frac{k-2}{4}\rfloor + (n - i) + 2\lfloor\frac{n-i-1}{4}\rfloor$, the corresponding polynomial is

$$D_{15}(x) = \sum_{k=\frac{n+1}{2}+1}^{n-2} \sum_{i=k+2}^n x^{k+2\lfloor\frac{k-2}{4}\rfloor+(n-i)+2\lfloor\frac{n-i-1}{4}\rfloor}.$$

Now, since the order of L_n^* is $2n$ and its size is $3n - 1$ and adding the polynomials obtained from all the cases, we get the required result. ■

Corollary 3: Taking the derivative of $D^*(L_n^*; x)$ given in Theorem 2 with respect to x and for $x = 1$ we get the restricted detour index of L_n^* . ■

Theorem 4: For even, $n \geq 6$, we have

$$D^*(L_n^*; x) = 2n + (3n - 1)x + 2x^2 + \sum_{i=2}^{n-2} x^{n+2\lfloor\frac{i-2}{4}\rfloor+2\lfloor\frac{n-i-2}{4}\rfloor+1} + \sum_{i=1}^{n-1} x^{(n-1)+2\lfloor\frac{i-3}{4}\rfloor+2\lfloor\frac{n-i-3}{4}\rfloor} + G(x),$$

where

$$G(x) = \begin{cases} G_1(x) & \text{for } i = \frac{n}{2} \equiv 0(\text{mod } 4) \\ G_2(x) & \text{for } i = \frac{n}{2} \equiv 1(\text{mod } 4) \\ G_3(x) & \text{for } i = \frac{n}{2} \equiv 2(\text{mod } 4) \\ G_4(x) & \text{for } i = \frac{n}{2} \equiv 3(\text{mod } 4), \end{cases}$$

in which

$$\begin{aligned} G_1(x) &= 2 \sum_{i \in S_{11}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k-1}{4}\rfloor+1} + 2 \sum_{i \in S_{12}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k-1}{4}\rfloor+1} + \\ &2 \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k-1}{4}\rfloor+1} + 2 \sum_{i \in S_{11} \setminus \{1\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i)+2\lfloor\frac{k-i-1}{4}\rfloor} + \\ &2 \sum_{k=\frac{n}{2}+2}^{n-1} x^{(k-1)+2\lfloor\frac{k-2}{4}\rfloor} + 2x^{(n-1)+2\lfloor\frac{n-3}{4}\rfloor} + 2 \sum_{i \in S_{12}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+2\lfloor\frac{k-i-1}{4}\rfloor} + \\ &\sum_{i=1}^{\frac{n}{2}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k}{4}\rfloor+2} + \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k}{4}\rfloor+2} \\ &\sum_{i=2}^{\frac{n}{2}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+1+2\lfloor\frac{k-i}{4}\rfloor} + 2 \sum_{k=\frac{n}{2}+1}^{n-1} x^{k+2\lfloor\frac{k-1}{4}\rfloor} + x^{n+2\lfloor\frac{n-2}{4}\rfloor} + \\ &\sum_{k=1}^{\frac{n}{2}} \sum_{i=k+2}^{\frac{n}{2}+(k-1)} x^{k+2\lfloor\frac{k-2}{4}\rfloor+(n-i)+2\lfloor\frac{n-i-1}{4}\rfloor} + \sum_{k=\frac{n}{2}+1}^{n-2} \sum_{i=k+2}^n x^{k+2\lfloor\frac{k-2}{4}\rfloor+(n-i)+2\lfloor\frac{n-i-1}{4}\rfloor} + \end{aligned}$$

$$\begin{aligned}
 & \sum_{k=2}^{\frac{n}{2}} \sum_{i=\frac{n}{2}+k}^n x^{(i-k)+1+2\lfloor \frac{i-k}{4} \rfloor}, \\
 G_2(x) = & 2 \sum_{i \in S_{21}}^{\frac{n}{2}+(i-1)} \sum_{k=i+2}^n x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k-1}{4} \rfloor+1} + 2 \sum_{i \in S_{22}}^{\frac{n}{2}+i} \sum_{k=i+2}^n x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k-1}{4} \rfloor+1} + \\
 & 2 \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k-1}{4} \rfloor+1} + 2 \sum_{i \in S_{21}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+2\lfloor \frac{k-i-1}{4} \rfloor} + 2 \sum_{k=\frac{n}{2}+2}^{n-1} x^{(k-1)+2\lfloor \frac{k-2}{4} \rfloor} + \\
 & 2x^{(n-1)+2\lfloor \frac{n-3}{4} \rfloor} + 2 \sum_{i \in S_{22} \setminus \{1, \frac{n}{2}\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i)+2\lfloor \frac{k-i-1}{4} \rfloor} + \sum_{i \in S_{23}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k}{4} \rfloor+2} + \\
 & \sum_{i \in S_{24}}^{\frac{n}{2}+(i-1)} \sum_{k=i+2}^n x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k}{4} \rfloor+2} + \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k}{4} \rfloor+2} + x^{n+2\lfloor \frac{n-2}{4} \rfloor} + \\
 & \sum_{i \in S_{23} \setminus \{1, \frac{n}{2}\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i)+1+2\lfloor \frac{k-i}{4} \rfloor} + \sum_{i \in S_{24}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+1+2\lfloor \frac{k-i}{4} \rfloor} + \sum_{k=\frac{n}{2}+2}^{n-1} x^{k+2\lfloor \frac{k-1}{4} \rfloor} + \\
 & \sum_{k \in S_{25}} \sum_{i=k+2}^{\frac{n}{2}+k} x^{k+2\lfloor \frac{k-2}{4} \rfloor+(n-i)+2\lfloor \frac{n-i-1}{4} \rfloor} + \sum_{k \in S_{26}} \sum_{i=k+2}^{\frac{n}{2}+(k-1)} x^{k+2\lfloor \frac{k-2}{4} \rfloor+(n-i)+2\lfloor \frac{n-i-1}{4} \rfloor} + \\
 & \sum_{k=\frac{n}{2}+1}^{n-2} \sum_{i=k+2}^n x^{k+2\lfloor \frac{k-2}{4} \rfloor+(n-i)+2\lfloor \frac{n-i-1}{4} \rfloor} + \sum_{k \in S_{25}} \sum_{i=\frac{n}{2}+k+1}^n x^{(i-k)+1+2\lfloor \frac{i-k}{4} \rfloor} + \sum_{i=\frac{n}{2}+1}^{n-1} x^{i+2\lfloor \frac{i-1}{4} \rfloor} + \\
 & \sum_{k \in S_{26} \setminus \{1\}} \sum_{i=\frac{n}{2}+k}^n x^{(i-k)+1+2\lfloor \frac{i-k}{4} \rfloor}, \\
 G_3(x) = & 2 \sum_{i=1}^{\frac{n}{2}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k-1}{4} \rfloor+1} + 2 \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k-1}{4} \rfloor+1} + \\
 & 2 \sum_{i=2}^{\frac{n}{2}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+2\lfloor \frac{k-i-1}{4} \rfloor} + 2 \sum_{k=\frac{n}{2}+1}^{n-1} x^{(k-1)+2\lfloor \frac{k-2}{4} \rfloor} + 2x^{(n-1)+2\lfloor \frac{n-3}{4} \rfloor} + \\
 & \sum_{i \in S_{31}}^{\frac{n}{2}+(i-1)} \sum_{k=i+2}^n x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k}{4} \rfloor+2} + \sum_{i \in S_{32}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k}{4} \rfloor+2} + \\
 & \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor \frac{i-1}{4} \rfloor+(n-k)+2\lfloor \frac{n-k}{4} \rfloor+2} + \sum_{i \in S_{31}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+1+2\lfloor \frac{k-i}{4} \rfloor} + \sum_{k=\frac{n}{2}+2}^{n-1} x^{k+2\lfloor \frac{k-1}{4} \rfloor} + \\
 & \sum_{i \in S_{32} \setminus \{1, \frac{n}{2}\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i)+1+2\lfloor \frac{k-i}{4} \rfloor} + x^{n+2\lfloor \frac{n-2}{4} \rfloor} + \sum_{k \in S_{33}} \sum_{i=k+2}^{\frac{n}{2}+k} x^{k+2\lfloor \frac{k-2}{4} \rfloor+(n-i)+2\lfloor \frac{n-i-1}{4} \rfloor} +
 \end{aligned}$$

$$\sum_{k \in S_{34}} \sum_{i=k+2}^{\frac{n}{2}+(k-i)} x^{k+2 \lfloor \frac{k-2}{4} \rfloor + (n-i)+2 \lfloor \frac{n-i-1}{4} \rfloor} + \sum_{k=\frac{n}{2}+1}^{n-2} \sum_{i=k+2}^n x^{k+2 \lfloor \frac{k-2}{4} \rfloor + (n-i)+2 \lfloor \frac{n-i-1}{4} \rfloor} +$$

$$\sum_{k \in S_{33}} \sum_{i=\frac{n}{2}+k+1}^n x^{(i-k)+1+2 \lfloor \frac{i-k}{4} \rfloor} + \sum_{k \in S_{34}} \sum_{i=\frac{n}{2}+k}^n x^{(i-k)+1+2 \lfloor \frac{i-k}{4} \rfloor} - x^{n+2 \lfloor \frac{n-1}{4} \rfloor},$$

and

$$G_4(x) = 2 \sum_{i \in S_{41}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k)+2 \lfloor \frac{n-k-1}{4} \rfloor + 1} + 2 \sum_{i \in S_{42}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k)+2 \lfloor \frac{n-k-1}{4} \rfloor + 1} +$$

$$2 \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k)+2 \lfloor \frac{n-k-1}{4} \rfloor + 1} + 2 \sum_{i \in S_{41} \setminus \{1\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i)+2 \lfloor \frac{k-i-1}{4} \rfloor} + 2x^{(n-1)+2 \lfloor \frac{n-3}{4} \rfloor} +$$

$$2 \sum_{k=\frac{n}{2}+2}^{n-1} x^{(k-1)+2 \lfloor \frac{k-2}{4} \rfloor} + 2 \sum_{i \in S_{42}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+2 \lfloor \frac{k-i-1}{4} \rfloor} + \sum_{i \in S_{43}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k)+2 \lfloor \frac{n-k}{4} \rfloor + 2} +$$

$$\sum_{i \in S_{44}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k)+2 \lfloor \frac{n-k}{4} \rfloor + 2} + \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k)+2 \lfloor \frac{n-k}{4} \rfloor + 2} + x^{n+2 \lfloor \frac{n-2}{4} \rfloor} +$$

$$\sum_{i \in S_{43}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+1+2 \lfloor \frac{k-i}{4} \rfloor} + \sum_{i \in S_{44} \setminus \{1, \frac{n}{2}\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i)+1+2 \lfloor \frac{k-i}{4} \rfloor} + \sum_{k=\frac{n}{2}+2}^{n-1} x^{k+2 \lfloor \frac{k-1}{4} \rfloor} +$$

$$\sum_{k \in S_{45}} \sum_{i=k+2}^{\frac{n}{2}+(k-1)} x^{k+2 \lfloor \frac{k-2}{4} \rfloor + (n-i)+2 \lfloor \frac{n-i-1}{4} \rfloor} + \sum_{k \in S_{46}} \sum_{i=k+2}^{\frac{n}{2}+k} x^{k+2 \lfloor \frac{k-2}{4} \rfloor + (n-i)+2 \lfloor \frac{n-i-1}{4} \rfloor} +$$

$$\sum_{k=\frac{n}{2}+1}^{n-2} \sum_{i=k+2}^n x^{k+2 \lfloor \frac{k-2}{4} \rfloor + (n-i)+2 \lfloor \frac{n-i-1}{4} \rfloor} + \sum_{k \in S_{46} \setminus \{\frac{n}{2}\}} \sum_{i=\frac{n}{2}+k+1}^n x^{(i-k)+1+2 \lfloor \frac{i-k}{4} \rfloor} - x^{n+2 \lfloor \frac{n-1}{4} \rfloor} +$$

$$\sum_{k \in S_{45}} \sum_{i=\frac{n}{2}+k}^n x^{(i-k)+1+2 \lfloor \frac{i-k}{4} \rfloor}.$$

Proof:

We refer to Figure 2, four main cases can be distinguished for finding $G(x)$ in which $G(x)$ is the polynomial corresponding to all possibilities of u_i, u_k and u_i, u'_k .

Case I: $\frac{n}{2} \equiv 0 \pmod{4}$

Denote $S_{11} = \{i : 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 1 \pmod{4} \text{ or } i \equiv 2 \pmod{4}\}$ and

$S_{12} = \{i : 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 0 \pmod{4} \text{ or } i \equiv 3 \pmod{4}\}.$

(1) Let u_i and u_k be two non-adjacent vertices of $V(L_n^*)$. Then the following subcases are considered:

(a) If $i \in S_{11}$ and $i + 2 \leq k \leq \frac{n}{2} + i$ or $i \in S_{12}$ and $i + 2 \leq k \leq \frac{n}{2} + (i - 1)$, or if

$\frac{n}{2} + 1 \leq i \leq n - 2$ and $i + 2 \leq k \leq n$, then by Theorem 2 Case III (1) and Case V (1), we get

$$D^*(u_i, u_k) = i + 2 \left\lfloor \frac{i-1}{4} \right\rfloor + (n-k) + 2 \left\lfloor \frac{n-k-1}{4} \right\rfloor + 1.$$

From the symmetry of L_n^* graph, $D^*(u_i, u_k) = D^*(u'_{n-i+1}, u'_{n-k+1})$, the corresponding polynomials are

$$D_1(x) = 2 \sum_{i \in S_{11}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k-1}{4} \rfloor + 1},$$

$$D_2(x) = 2 \sum_{i \in S_{12}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k-1}{4} \rfloor + 1},$$

$$D_3(x) = 2 \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k-1}{4} \rfloor + 1}.$$

(b) If $i \in S_{11} \setminus \{1\}$ and $\frac{n}{2} + i + 1 \leq k \leq n$ or $i = 1$ and $\frac{n}{2} + 2 \leq k \leq n - 1$ or if

$i \in S_{12}$ and $\frac{n}{2} + i \leq k \leq n$ then by Theorem 2 case I (1), case II (1) and case IV (1) we get

$$D^*(u_i, u_k) = (k - i) + 2 \lfloor \frac{k-i-1}{4} \rfloor \text{ and } D^*(u_1, u_k) = (k - 1) + 2 \lfloor \frac{k-2}{4} \rfloor \text{ and if } i = 1 \text{ and } k = n \text{ then}$$

$$D^*(u_1, u_n) = (n - 1) + 2 \lfloor \frac{n-3}{4} \rfloor.$$

From the symmetry of L_n^* , $D^*(u_i, u_k) = D^*(u'_{n-i+1}, u'_{n-k+1})$, the corresponding polynomials are:

$$D_4(x) = 2 \sum_{i \in S_{11} \setminus \{1\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i) + 2 \lfloor \frac{k-i-1}{4} \rfloor},$$

$$D_5(x) = 2 \sum_{k=\frac{n}{2}+2}^{n-1} x^{(k-1) + 2 \lfloor \frac{k-2}{4} \rfloor},$$

$$D_6(x) = 2x^{(n-1) + 2 \lfloor \frac{n-3}{4} \rfloor}$$

$$D_7(x) = 2 \sum_{i \in S_{12}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i) + 2 \lfloor \frac{k-i-1}{4} \rfloor}.$$

(2) For the restricted detour between u_i and u'_k :

(a) If $1 \leq i \leq \frac{n}{2}$ and $i + 2 \leq k \leq \frac{n}{2} + (i - 1)$ or if $\frac{n}{2} + 1 \leq i \leq n - 2$ and $i + 2 \leq k \leq n$

then similarly from Theorem 2 Case III (2), Case V (2) we get $D^*(u_i, u'_k) = i + 2 \lfloor \frac{i-1}{4} \rfloor + (n - k) + 2 \lfloor \frac{n-k}{4} \rfloor + 2$.

This produce the polynomials

$$D_8(x) = \sum_{i=1}^{\frac{n}{2}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k}{4} \rfloor + 2},$$

$$D_9(x) = \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k}{4} \rfloor + 2}.$$

(b) If $2 \leq i \leq \frac{n}{2}$ and $\frac{n}{2} + i \leq k \leq n$ or $i = 1$ and $\frac{n}{2} + 1 \leq k \leq n - 1$ or $i = 1$ and $k = n$ then from Theorem 2 Case I (2), Case II (2), Case IV (2) we get :

$$D^*(u_i, u'_k) = (k - i) + 1 + 2 \lfloor \frac{k-i}{4} \rfloor,$$

$$D^*(u_1, u'_k) = k + 2 \lfloor \frac{k-1}{4} \rfloor,$$

$$D^*(u_1, u'_n) = n + 2 \lfloor \frac{n-2}{4} \rfloor.$$

Evidently, $D^*(u_1, u'_k) = D^*(u'_1, u_k)$ where $\frac{n}{2} + i \leq k \leq n - 1$ the polynomials are

$$D_{10}(x) = \sum_{i=2}^{\frac{n}{2}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+1+2\lfloor \frac{k-i}{4} \rfloor},$$

$$D_{11}(x) = 2 \sum_{k=\frac{n}{2}+1}^{n-1} x^{k+2\lfloor \frac{k-1}{4} \rfloor},$$

$$D_{12}(x) = x^{n+2\lfloor \frac{n-2}{4} \rfloor}.$$

(c) If $1 \leq k \leq \frac{n}{2}$ and $k + 2 \leq i \leq \frac{n}{2} + (k - 1)$ or $\frac{n}{2} + 1 \leq k \leq n - 2$ and $k + 2 \leq i \leq n$ then from Theorem 2 Case VII (1) and (2) the restricted detour distance is

$$D^*(u'_k, u_i) = k + 2 \left\lfloor \frac{k-2}{4} \right\rfloor + (n-i) + 2 \left\lfloor \frac{n-i-1}{4} \right\rfloor,$$

and the corresponding polynomials are

$$D_{13}(x) = \sum_{k=1}^{\frac{n}{2}} \sum_{i=k+2}^{\frac{n}{2}+(k-1)} x^{k+2\lfloor \frac{k-2}{4} \rfloor + (n-i) + 2\lfloor \frac{n-i-1}{4} \rfloor},$$

$$D_{14}(x) = \sum_{k=\frac{n}{2}+1}^{n-2} \sum_{i=k+2}^n x^{k+2\lfloor \frac{k-2}{4} \rfloor + (n-i) + 2\lfloor \frac{n-i-1}{4} \rfloor}.$$

(d) If $2 \leq k \leq \frac{n}{2}$ and $\frac{n}{2} + k \leq i \leq n$ then from Figure 2 we notice that between u'_k and u_i where $2 \leq k \leq \frac{n}{2}$ and $\frac{n}{2} + k \leq i \leq n$ there is a restricted detour path P, namely

$P : u'_k, u_k, u_{k+1}, u_{k+2}, u'_{k+2}, u'_{k+3}, u'_{k+4}, \dots, u'_i, u_i$ (or \dots, u_{i-1}, u_i) which is of the length

$$(i - k) + 1 + 2 \left\lfloor \frac{i - k}{4} \right\rfloor.$$

The polynomial is

$$D_{15}(x) = \sum_{k=2}^{\frac{n}{2}} \sum_{i=\frac{n}{2}+k}^n x^{(i-k)+1+2\lfloor \frac{i-k}{4} \rfloor}.$$

Case II : $\frac{n}{2} \equiv 1 \pmod{4}$

Denote $S_{21} = \{ i : 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 0 \pmod{4} \}$,

$S_{22} = \{ i : 1 \leq i \leq \frac{n}{2} \text{ and } i \not\equiv 0 \pmod{4} \}$,

$S_{23} = \{ i : 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 1 \pmod{4} \}$,

$S_{24} = \{ i : 1 \leq i \leq \frac{n}{2} \text{ and } i \not\equiv 1 \pmod{4} \}$,

$S_{25} = \{ k : 1 \leq k \leq \frac{n}{2} \text{ and } k \equiv 3 \pmod{4} \}$,

$S_{26} = \{ k : 1 \leq k \leq \frac{n}{2} \text{ and } k \not\equiv 3 \pmod{4} \}$.

(I) Let u_i and u_k be two non-adjacent vertices in $V(L_n^*)$. Then the following subcases are considered:

(a) If $i \in S_{21}$ and $i + 2 \leq k \leq \frac{n}{2} + (i - 1)$ or $i \in S_{22}$ and $i + 2 \leq k \leq \frac{n}{2} + i$ or $\frac{n}{2} + 1 \leq i \leq n - 2$ and $i + 2 \leq k \leq n$ then by Theorem 2 Case III (1) and Case V (1), we get

$$D^*(u_i, u_k) = i + 2 \left\lfloor \frac{i-1}{4} \right\rfloor + (n-k) + 2 \left\lfloor \frac{n-k-1}{4} \right\rfloor + 1.$$

Evidently, $D^*(u_i, u_k) = D^*(u'_{n-i+1}, u'_{n-k+1})$ and then we get the following polynomials

$$D_1(x) = 2 \sum_{i \in S_{21}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k-1}{4} \rfloor + 1},$$

$$D_2(x) = 2 \sum_{i \in S_{22}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k-1}{4} \rfloor + 1},$$

$$D_3(x) = 2 \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k-1}{4} \rfloor + 1}.$$

(b) If $i \in S_{21}$ and $\frac{n}{2} + i \leq k \leq n$ or $i = 1$ and $\frac{n}{2} + 2 \leq k \leq n - 1$ or $i \in S_{22} \setminus \{1, \frac{n}{2}\}$ and $\frac{n}{2} + i + 1 \leq k \leq n$ then $D^*(u_i, u_k) = (k - i) + 2 \lfloor \frac{k-i-1}{4} \rfloor$ and $D^*(u_1, u_k) = (k - 1) + 2 \lfloor \frac{k-2}{4} \rfloor$ and if $i = 1$ and $k = n$ then $D^*(u_1, u_n) = (n - 1) + 2 \lfloor \frac{n-3}{4} \rfloor$.

By similar way we get the following polynomials

$$D_4(x) = 2 \sum_{i \in S_{21}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i) + 2 \lfloor \frac{k-i-1}{4} \rfloor},$$

$$D_5(x) = 2 \sum_{k=\frac{n}{2}+2}^{n-1} x^{(k-1) + 2 \lfloor \frac{k-2}{4} \rfloor},$$

$$D_6(x) = 2x^{(n-1) + 2 \lfloor \frac{n-3}{4} \rfloor},$$

$$D_7(x) = 2 \sum_{i \in S_{22} \setminus \{1, \frac{n}{2}\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i) + 2 \lfloor \frac{k-i-1}{4} \rfloor}.$$

(2) For the restricted detour distance between u_i and u'_k , similarly, we have

(a) If $i \in S_{23}$ and $i + 2 \leq k \leq \frac{n}{2} + i$ or $i \in S_{24}$ and $i + 2 \leq k \leq \frac{n}{2} + (i - 1)$ or $\frac{n}{2} + 1 \leq i \leq n - 2$ and $i + 2 \leq k \leq n$ then $D^*(u_i, u'_k) = i + 2 \lfloor \frac{i-1}{4} \rfloor + (n - k) + 2 \lfloor \frac{n-k}{4} \rfloor + 2$.

Thus, we have the following polynomials:

$$D_8(x) = \sum_{i \in S_{23}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k}{4} \rfloor + 2},$$

$$D_9(x) = \sum_{i \in S_{24}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k}{4} \rfloor + 2},$$

$$D_{10}(x) = \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k}{4} \rfloor + 2}.$$

(b) If $i \in S_{23} \setminus \{1, \frac{n}{2}\}$ and $\frac{n}{2} + i + 1 \leq k \leq n$ or $i \in S_{24}$ and $\frac{n}{2} + i \leq k \leq n$ or $i = 1$ and $\frac{n}{2} + 2 \leq k \leq n - 1$, then similarly the restricted detour distance are

$$D^*(u_i, u'_k) = (k - i) + 1 + 2 \lfloor \frac{k-i}{4} \rfloor, D^*(u_1, u'_k) = k + 2 \lfloor \frac{k-1}{4} \rfloor$$

$$\text{and } D^*(u_1, u'_n) = n + 2 \lfloor \frac{n-2}{4} \rfloor,$$

the corresponding polynomials are

$$D_{11}(x) = \sum_{i \in S_{23} \setminus \{1, \frac{n}{2}\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i)+1+2\lfloor \frac{k-i}{4} \rfloor},$$

$$D_{12}(x) = \sum_{i \in S_{24}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+1+2\lfloor \frac{k-i}{4} \rfloor},$$

$$D_{13}(x) = \sum_{k=\frac{n}{2}+2}^{n-1} x^{k+2\lfloor \frac{k-1}{4} \rfloor},$$

$$D_{14}(x) = x^{n+2\lfloor \frac{n-2}{4} \rfloor}.$$

(c) If $k \in S_{25}$ and $k + 2 \leq i \leq \frac{n}{2} + k$ or $k \in S_{26}$ and $k + 2 \leq i \leq \frac{n}{2} + (k - 1)$ or $\frac{n}{2} + 1 \leq k \leq n - 2$ and $k + 2 \leq i \leq n$ then by the same way

$$D^*(u'_k, u_i) = k + 2 \left\lfloor \frac{k-2}{4} \right\rfloor + (n-i) + 2 \left\lfloor \frac{n-i-1}{4} \right\rfloor,$$

and the polynomials are

$$D_{15}(x) = \sum_{k \in S_{25}} \sum_{i=k+2}^{\frac{n}{2}+k} x^{k+2\lfloor \frac{k-2}{4} \rfloor + (n-i) + 2\lfloor \frac{n-i-1}{4} \rfloor},$$

$$D_{16}(x) = \sum_{k \in S_{26}} \sum_{i=k+2}^{\frac{n}{2}+(k-1)} x^{k+2\lfloor \frac{k-2}{4} \rfloor + (n-i) + 2\lfloor \frac{n-i-1}{4} \rfloor},$$

$$D_{17}(x) = \sum_{k=\frac{n}{2}+1}^{n-2} \sum_{i=k+2}^n x^{k+2\lfloor \frac{k-2}{4} \rfloor + (n-i) + 2\lfloor \frac{n-i-1}{4} \rfloor}.$$

(d) If $k \in S_{25}$ and $\frac{n}{2} + k + 1 \leq i \leq n$ or $k \in S_{26}$ and $\frac{n}{2} + k \leq i \leq n$ or $k = 1$ and $\frac{n}{2} + 1 \leq i \leq n - 1$, then there is a restricted detour path as in Case I (d) above, we have the following polynomials:

$$D_{18}(x) = \sum_{k \in S_{25}} \sum_{i=\frac{n}{2}+k+1}^n x^{(i-k)+1+2\lfloor \frac{i-k}{4} \rfloor},$$

$$D_{19}(x) = \sum_{k \in S_{26} \setminus \{1\}} \sum_{i=\frac{n}{2}+k}^n x^{(i-k)+1+2\lfloor \frac{i-k}{4} \rfloor},$$

$$D_{20}(x) = \sum_{i=\frac{n}{2}+1}^{n-1} x^{i+2\lfloor \frac{i-1}{4} \rfloor}.$$

Case III : $\frac{n}{2} \equiv 2 \pmod{4}$

Denote $S_{31} = \{ i : 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 0 \pmod{4} \text{ or } i \equiv 3 \pmod{4} \}$,

$S_{32} = \{ i : 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 1 \pmod{4} \text{ or } i \equiv 2 \pmod{4} \}$,

$S_{33} = \{ k : 1 \leq k \leq \frac{n}{2} \text{ and } k \equiv 0 \pmod{4} \text{ or } k \equiv 3 \pmod{4} \}$,

$S_{34} = \{ k : 1 \leq k \leq \frac{n}{2} \text{ and } k \equiv 1 \pmod{4} \text{ or } k \equiv 2 \pmod{4} \}$.

(1) Also for two non-adjacent vertices u_i and u_k in $V(L_n^*)$, we have the following subcases:

(a) If $1 \leq i \leq \frac{n}{2}$ and $i + 2 \leq k \leq \frac{n}{2} + (i - 1)$ or $\frac{n}{2} + 1 \leq i \leq n - 2$ and $i + 2 \leq k \leq n$

then $D^*(u_i, u_k) = i + 2 \left\lfloor \frac{i-1}{4} \right\rfloor + (n - k) + 2 \left\lfloor \frac{n-k-1}{4} \right\rfloor + 1,$

and we get the following polynomials:

$$D_1(x) = 2 \sum_{i=1}^{\frac{n}{2}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k-1}{4}\rfloor+1},$$

$$D_2(x) = 2 \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k-1}{4}\rfloor+1}.$$

(b) If $2 \leq i \leq \frac{n}{2}$ and $\frac{n}{2} + i \leq k \leq n$ or $i = 1$ and $\frac{n}{2} + 1 \leq k \leq n - 1$ then

$$D^*(u_i, u_k) = (k - i) + 2 \lfloor\frac{k-i-1}{4}\rfloor \text{ and } D^*(u_1, u_k) = (k - 1) + 2 \lfloor\frac{k-2}{4}\rfloor \text{ and if } i = 1 \text{ and } k = n \text{ then}$$

$$D^*(u_1, u_n) = (n - 1) + 2 \lfloor\frac{n-3}{4}\rfloor,$$

the corresponding polynomials are:

$$D_3(x) = 2 \sum_{i=2}^{\frac{n}{2}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+2\lfloor\frac{k-i-1}{4}\rfloor},$$

$$D_4(x) = 2 \sum_{k=\frac{n}{2}+1}^{n-1} x^{(k-1)+2\lfloor\frac{k-2}{4}\rfloor},$$

$$D_5(x) = 2x^{(n-1)+2\lfloor\frac{n-3}{4}\rfloor}.$$

(2) For the restricted detour distance between u_i and u'_k the subcases are:

(a) If $i \in S_{31}$ and $i + 2 \leq k \leq \frac{n}{2} + (i - 1)$ or $i \in S_{32}$ and $i + 2 \leq k \leq \frac{n}{2} + i$ or $\frac{n}{2} + 1 \leq i \leq n - 2$ and $i + 2 \leq k \leq n$ then $D^*(u_i, u'_k) = i + 2 \lfloor\frac{i-1}{4}\rfloor + (n - k) + 2 \lfloor\frac{n-k}{4}\rfloor + 2,$

and the polynomials are:

$$D_6(x) = \sum_{i \in S_{31}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k}{4}\rfloor+2},$$

$$D_7(x) = \sum_{i \in S_{32}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k}{4}\rfloor+2},$$

$$D_8(x) = \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k}{4}\rfloor+2}.$$

(b) If $i \in S_{31}$ and $\frac{n}{2} + i \leq k \leq n$ or $i \in S_{32} \setminus \{1, \frac{n}{2}\}$ and $\frac{n}{2} + i + 1 \leq k \leq n$ or $i = 1$ and $\frac{n}{2} + 2 \leq k \leq n - 1,$ then $D^*(u_i, u'_k) = (k - i) + 1 + 2 \lfloor\frac{k-i}{4}\rfloor, D^*(u_1, u'_k) = k + 2 \lfloor\frac{k-1}{4}\rfloor$

$$\text{and } D^*(u_1, u'_n) = n + 2 \lfloor\frac{n-2}{4}\rfloor.$$

Thus the corresponding polynomials are:

$$D_9(x) = \sum_{i \in S_{31}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+1+2\lfloor\frac{k-i}{4}\rfloor},$$

$$D_{10}(x) = \sum_{i \in S_{32} \setminus \{1, \frac{n}{2}\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i)+1+2\lfloor\frac{k-i}{4}\rfloor},$$

$$D_{11}(x) = \sum_{k=\frac{n}{2}+2}^{n-1} x^{k+2 \lfloor \frac{k-1}{4} \rfloor},$$

$$D_{12}(x) = x^{n+2 \lfloor \frac{n-2}{4} \rfloor}.$$

(c) If $k \in S_{33}$ and $k + 2 \leq i \leq \frac{n}{2} + k$ or $k \in S_{34}$ and $k + 2 \leq i \leq \frac{n}{2} + (k - 1)$ or $\frac{n}{2} + 1 \leq k \leq n - 2$ and $k + 2 \leq i \leq n$ then $D^*(u'_k, u_i) = k + 2 \lfloor \frac{k-2}{4} \rfloor + (n - i) + 2 \lfloor \frac{n-i-1}{4} \rfloor$, and we get the following polynomials

$$D_{13}(x) = \sum_{k \in S_{33}} \sum_{i=k+2}^{\frac{n}{2}+k} x^{k+2 \lfloor \frac{k-2}{4} \rfloor + (n-i) + 2 \lfloor \frac{n-i-1}{4} \rfloor},$$

$$D_{14}(x) = \sum_{k \in S_{34}} \sum_{i=k+2}^{\frac{n}{2}+(k-1)} x^{k+2 \lfloor \frac{k-2}{4} \rfloor + (n-i) + 2 \lfloor \frac{n-i-1}{4} \rfloor},$$

$$D_{15}(x) = \sum_{k=\frac{n}{2}+1}^{n-2} \sum_{i=k+2}^n x^{k+2 \lfloor \frac{k-2}{4} \rfloor + (n-i) + 2 \lfloor \frac{n-i-1}{4} \rfloor}.$$

(d) If $k \in S_{33}$ and $\frac{n}{2} + k + 1 \leq i \leq n$ or $k \in S_{34} \setminus \{1\}$ and $\frac{n}{2} + k \leq i \leq n$ or $k = 1$ and $\frac{n}{2} + 1 \leq i \leq n - 1$, then the corresponding polynomials are:

$$D_{16}(x) = \sum_{k \in S_{33}} \sum_{i=\frac{n}{2}+k+1}^n x^{(i-k)+1+2 \lfloor \frac{i-k}{4} \rfloor},$$

$$D_{17}(x) = \sum_{k \in S_{34}} \sum_{i=\frac{n}{2}+k}^n x^{(i-k)+1+2 \lfloor \frac{i-k}{4} \rfloor} - x^{n+2 \lfloor \frac{n-1}{4} \rfloor}.$$

Case IV : $\frac{n}{2} \equiv 3 \pmod{4}$

- Let $S_{41} = \{ i : 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 1 \pmod{4} \}$,
 $S_{42} = \{ i : 1 \leq i \leq \frac{n}{2} \text{ and } i \not\equiv 1 \pmod{4} \}$,
 $S_{43} = \{ i : 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 0 \pmod{4} \}$,
 $S_{44} = \{ i : 1 \leq i \leq \frac{n}{2} \text{ and } i \not\equiv 0 \pmod{4} \}$,
 $S_{45} = \{ k : 1 \leq k \leq \frac{n}{2} \text{ and } k \equiv 2 \pmod{4} \}$,
 $S_{46} = \{ k : 1 \leq k \leq \frac{n}{2} \text{ and } k \not\equiv 2 \pmod{4} \}$.

(1) Let u_i and u_k be two non-adjacent vertices in $V(L_n^*)$. Then the following subcases are considered:

(a) If $i \in S_{41}$ and $i + 2 \leq k \leq \frac{n}{2} + i$ or $i \in S_{42}$ and $i + 2 \leq k \leq \frac{n}{2} + (i - 1)$ or $\frac{n}{2} + 1 \leq i \leq n - 2$ and $i + 2 \leq k \leq n$ then $D^*(u_i, u_k) = i + 2 \lfloor \frac{i-1}{4} \rfloor + (n - k) + 2 \lfloor \frac{n-k-1}{4} \rfloor + 1$,

and we get the following polynomials

$$D_1(x) = 2 \sum_{i \in S_{41}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k-1}{4} \rfloor + 1},$$

$$D_2(x) = 2 \sum_{i \in S_{42}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2 \lfloor \frac{i-1}{4} \rfloor + (n-k) + 2 \lfloor \frac{n-k-1}{4} \rfloor + 1},$$

$$D_3(x) = 2 \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k-1}{4}\rfloor+1}.$$

(b) If $i \in S_{41} \setminus \{1\}$ and $\frac{n}{2} + i + 1 \leq k \leq n$ or $i = 1$ and $\frac{n}{2} + 2 \leq k \leq n - 1$ or

$i \in S_{42}$ and $\frac{n}{2} + i \leq k \leq n$ then $D^*(u_i, u_k) = (k - i) + 2\lfloor\frac{k-i-1}{4}\rfloor$ and $D^*(u_1, u_k) = (k - 1) + 2\lfloor\frac{k-2}{4}\rfloor$ and if $i = 1$ and $k = n$ then $D^*(u_1, u_n) = (n - 1) + 2\lfloor\frac{n-3}{4}\rfloor$,

and we get the following polynomials

$$D_4(x) = 2 \sum_{i \in S_{41} \setminus \{1\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i)+2\lfloor\frac{k-i-1}{4}\rfloor},$$

$$D_5(x) = 2 \sum_{k=\frac{n}{2}+2}^{n-1} x^{(k-1)+2\lfloor\frac{k-2}{4}\rfloor},$$

$$D_6(x) = 2x^{(n-1)+2\lfloor\frac{n-3}{4}\rfloor},$$

$$D_7(x) = 2 \sum_{i \in S_{42}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+2\lfloor\frac{k-i-1}{4}\rfloor}.$$

(2) For the restricted detour distance between u_i and u'_k :

(a) If $i \in S_{43}$ and $i + 2 \leq k \leq \frac{n}{2} + (i - 1)$ or $i \in S_{44}$ and $i + 2 \leq k \leq \frac{n}{2} + i$ or $\frac{n}{2} + 1 \leq i \leq n - 2$ and $i + 2 \leq k \leq n$ then $D^*(u_i, u'_k) = i + 2\lfloor\frac{i-1}{4}\rfloor + (n - k) + 2\lfloor\frac{n-k}{4}\rfloor + 2$.

The corresponding polynomials are:

$$D_8(x) = \sum_{i \in S_{43}} \sum_{k=i+2}^{\frac{n}{2}+(i-1)} x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k}{4}\rfloor+2},$$

$$D_9(x) = \sum_{i \in S_{44}} \sum_{k=i+2}^{\frac{n}{2}+i} x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k}{4}\rfloor+2},$$

$$D_{10}(x) = \sum_{i=\frac{n}{2}+1}^{n-2} \sum_{k=i+2}^n x^{i+2\lfloor\frac{i-1}{4}\rfloor+(n-k)+2\lfloor\frac{n-k}{4}\rfloor+2}.$$

(b) If $i \in S_{43}$ and $\frac{n}{2} + i \leq k \leq n$ or $i \in S_{44} \setminus \{1, \frac{n}{2}\}$ and $\frac{n}{2} + i + 1 \leq k \leq n$ or $i = 1$ and $\frac{n}{2} + 2 \leq k \leq n - 1$, then $D^*(u_i, u'_k) = (k - i) + 1 + 2\lfloor\frac{k-i}{4}\rfloor$, $D^*(u_1, u'_k) = k + 2\lfloor\frac{k-1}{4}\rfloor$ and $D^*(u_1, u'_n) = n + 2\lfloor\frac{n-2}{4}\rfloor$,

and we have the following polynomials:

$$D_{11}(x) = \sum_{i \in S_{43}} \sum_{k=\frac{n}{2}+i}^n x^{(k-i)+1+2\lfloor\frac{k-i}{4}\rfloor},$$

$$D_{12}(x) = \sum_{i \in S_{44} \setminus \{1, \frac{n}{2}\}} \sum_{k=\frac{n}{2}+i+1}^n x^{(k-i)+1+2\lfloor\frac{k-i}{4}\rfloor},$$

$$D_{13}(x) = \sum_{k=\frac{n}{2}+2}^{n-1} x^{k+2\lfloor\frac{k-1}{4}\rfloor},$$

$$D_{14}(x) = x^{n+2\lfloor\frac{n-2}{4}\rfloor}.$$

(c) If $k \in S_{45}$ and $k + 2 \leq i \leq \frac{n}{2} + (k - 1)$ or $k \in S_{46}$ and $k + 2 \leq i \leq \frac{n}{2} + k$ or $\frac{n}{2} + 1 \leq k \leq n - 2$ and $k + 2 \leq i \leq n$ then $D^*(u'_k, u_i) = k + 2\lfloor\frac{k-2}{4}\rfloor + (n - i) + 2\lfloor\frac{n-i-1}{4}\rfloor$, and we get the following polynomials

$$D_{15}(x) = \sum_{k \in S_{45}} \sum_{i=k+2}^{\frac{n}{2}+(k-1)} x^{k+2\lfloor\frac{k-2}{4}\rfloor+(n-i)+2\lfloor\frac{n-i-1}{4}\rfloor},$$

$$D_{16}(x) = \sum_{k \in S_{46}} \sum_{i=k+2}^{\frac{n}{2}+k} x^{k+2\lfloor\frac{k-2}{4}\rfloor+(n-i)+2\lfloor\frac{n-i-1}{4}\rfloor},$$

$$D_{17}(x) = \sum_{k=\frac{n}{2}+1}^{n-2} \sum_{i=k+2}^n x^{k+2\lfloor\frac{k-2}{4}\rfloor+(n-i)+2\lfloor\frac{n-i-1}{4}\rfloor}.$$

(d) If $k \in S_{45}$ and $\frac{n}{2} + k \leq i \leq n$ or $k \in S_{46} \setminus \{1, \frac{n}{2}\}$ and $\frac{n}{2} + k + 1 \leq i \leq n$ or $k = 1$ and $\frac{n}{2} + 2 \leq i \leq n - 1$, then the corresponding polynomials are:

$$D_{18}(x) = \sum_{k \in S_{45}} \sum_{i=\frac{n}{2}+k}^n x^{(i-k)+1+2\lfloor\frac{i-k}{4}\rfloor},$$

$$D_{19}(x) = \sum_{k \in S_{46} \setminus \{\frac{n}{2}\}} \sum_{i=\frac{n}{2}+k+1}^n x^{(i-k)+1+2\lfloor\frac{i-k}{4}\rfloor} - x^{n+2\lfloor\frac{n-1}{4}\rfloor}.$$

Now, add the polynomials obtained from all the cases alone, we get $G_1(x)$, $G_2(x)$, $G_3(x)$ and $G_4(x)$ respectively.

Since, the order in L_n^* is $2n$, size is $(3n - 1)$ and $D^*(u_1, u'_2) = D^*(u_{n-1}, u'_n) = 2$,

$$D^*(u_i, u'_{i+1}) = n + 2\lfloor\frac{i-2}{4}\rfloor + 2\lfloor\frac{n-i-2}{4}\rfloor + 1 \quad \text{where } 2 \leq i \leq n - 2,$$

and

$$D^*(u'_i, u_{i+1}) = (n - 1) + 2\lfloor\frac{i-3}{4}\rfloor + 2\lfloor\frac{n-i-3}{4}\rfloor \quad \text{where } 1 \leq i \leq n - 1.$$

So, from Theorem 2, Case II (3) and Case (VI) (1), (2) we get the following corresponding polynomials:

$$D_{20}(x) = 2x^2,$$

$$D_{21}(x) = \sum_{i=2}^{n-2} x^{n+2\lfloor\frac{i-2}{4}\rfloor+2\lfloor\frac{n-i-2}{4}\rfloor+1},$$

$$D_{22}(x) = \sum_{i=1}^{n-1} x^{(n-1)+2\lfloor\frac{i-3}{4}\rfloor+2\lfloor\frac{n-i-3}{4}\rfloor}.$$

Finally, adding the polynomials we get the required result. ■

Remark 1: For small values of n we can obtain the restricted detour polynomials by direct calculation as given below:

$$D^*(L_2^*; x) = 4 + 5x + x^2,$$

$$D^*(L_3^*; x) = 6 + 8x + 6x^2 + x^3,$$

$$D^*(L_4^*; x) = 8 + 10x + 2x^2 + 5x^3 + 7x^4 + 3x^5,$$

$$D^*(L_5^*; x) = 10 + 14x + 2x^2 + x^3 + 10x^4 + 10x^5 + 8x^6.$$

Corollary 5. Taking the derivative of $D^*(L_n^*; x)$ given in Theorem 4 with respect to x and for $x = 1$, we get the restricted detour index of L_n^* . ■

In the next theorem we determine the restricted detour diameter of L_n^* .

Theorem 6. For $n \geq 3$, we have

$$Diam^*(L_n^*) = \begin{cases} (n-1) + 2 \left\lfloor \frac{n-3}{4} \right\rfloor & \text{if } n \equiv 0(\text{mod}4) \text{ or } n \equiv 1(\text{mod}4) \\ n + 2 \left\lfloor \frac{n-2}{4} \right\rfloor & \text{if } n \equiv 2(\text{mod}4) \text{ or } n \equiv 3(\text{mod}4). \end{cases}$$

Proof.

To find the restricted detour diameter of L_n^* we focus on the longest path P such that $V < P \geq P$ of L_n^* and for these we have two cases:

Case I : If $n \equiv 0(\text{mod}4)$ or $n \equiv 1(\text{mod}4)$ then the longest path in the graph L_n^* is between u_1 and u_n which is of the length $(n-1) + 2 \left\lfloor \frac{n-3}{4} \right\rfloor$.

Case II : If $n \equiv 2(\text{mod}4)$ or $n \equiv 3(\text{mod}4)$ then the longest path in the graph L_n^* is between u_1 and u'_n which is of the length $n + 2 \left\lfloor \frac{n-2}{4} \right\rfloor$.

Notice that these longest paths are not unique, as $D^*(u_1, u'_3) = D^*(u_1, u'_n)$ and $D^*(u_1, u_3) = D^*(u_1, u_n)$. ■

References

- [1] Abdullah, H. O., "Hosoya polynomials of Steiner distance of some graphs", Ph.D. Dissertation, Salahaddin University/Erbil, Iraq, (2007).
- [2] Ali, I. D., "Restricted detour polynomials of cycle related graphs", M.Sc. Thesis, Salahaddin University/Erbil, Iraq, (2014).
- [3] Abdullah, H. O. and Mohammed-Saleh, G. A., "Detour Hosoya polynomial of some compound graphs", Raf, J. Com. Sci. and Maths., Vol. 7, No. 1, pp. 199-204, (2010).
- [4] Ali, A. A. and Mohammed-Saleh, G. A., "The detour polynomials of ladder graphs", Raf. J. Math. Comput. Sci., Vol. 9, No. 1, pp. 139-146, (2012).
- [5] Ali, A. A. and Mohammed-Saleh, G. A., "The restricted detour polynomials of a hexagonal chain and a ladder graphs", J. Math. Comput. Sci., Vol. 2, No. 6, pp. 1622-1633, (2012).
- [6] Chartrand, G., John, G. L. and Tian, S., "Detour distance in graphs", Annals of discrete mathematics, Vol. 55, pp. 127-136, (1993).
- [7] Chartrand, G. and Lesniak, L., "Graphs and digraphs", 2nd edition, Wadsworth and Brooks/Cole, California, USA, (1986).
- [8] Chartrand, G. and Zhang, P., "Distance in graphs-taking long view", AKCEJ. Graphs Combin., Vol. 1, No. 1, pp. 1-13, (2004).
- [9] Harray, F., "Graph theory", Addison-Wesely, Reading, Massachusetts, (1969).
- [10] Mohammed-Saleh, G. A., "ON the detour distance and detour polynomials of graphs", Ph. D. Dissertation, Salahaddin University/Erbil, Iraq, (2013).

